

# On the Classification of $G$ -Graded Twisted Algebras over Finite Abelian Groups

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## Abstract

Let  $G$  be a group and let  $W$  be an algebra over a field  $K$ . We will say that  $W$  is a  $G$ -graded twisted algebra if  $W$  can be written as  $W = \oplus_{g \in G} W_g$  with  $W_a W_b \subset W_{ab}$ , and where each  $W_g$  is a one dimensional  $K$ -vector space. It is also assumed that  $W$  has no monomial which is a zero divisor which means that for each pair of nonzero elements  $w_a \in W_a$ ,  $w_b \in W_b$ ,  $w_a \cdot w_b \neq 0$ . We also demand that  $W$  has a multiplicative identity element. We focus in the case where  $G$  is a finite abelian group and  $K = \mathbb{C}$  or  $K = \mathbb{R}$ . In this article, using methods of group cohomology, we classify all associative  $G$ -graded twisted algebras in the case  $G$  is a finite abelian group. On the other hand, by generalizing some of the arguments developed in [1] we present a classification of all  $G$ -graded twisted algebras that satisfy certain symmetry condition.

## 1 Introduction

$G$ -graded twisted algebras were introduced in [2], and independently in [4], as distinguished mathematical structures which arise naturally in theoretical physics [5], [6], [7], [8], [9] and [3]. A  $G$ -graded twisted algebra  $W$  is an algebra over a commutative ring  $R$  with a  $G$ -grading, i.e.,  $W = \oplus_{g \in G} W_g$ , with  $W_a W_b \subset W_{ab}$ . Each  $W_g$  is assumed to be a free  $R$ -module of rank one, and we demand that  $W$  is free of zero monomial divisors, i.e.,  $w_a \cdot w_b \neq 0$  for every non-zero elements  $w_a \in W_a$ ,  $w_b \in W_b$ . We also demand that  $W$  has an identity element  $1 \in W_e$ , where  $W_e$  denotes the graded component corresponding to the identity element  $e \in G$ .

A classification of all associative  $G$ -graded twisted algebras over the real and complex numbers over finite cyclic groups is achieved in [1] by using standard techniques from group cohomology. In the first part of this article we generalize the methods developed in [1] in order to provide a classification of all associative  $G$ -graded twisted algebras over any finite abelian group. The exact number of distinct (graded) isomorphism classes is given in Theorem 4 where a formula to count them is presented.

The second part of this article deals with the classification problem for non-associative  $G$ -graded twisted algebras that satisfy a particular type of symmetry condition. In Theorem 20 we also provide an exact formula to count (up to graded isomorphisms) all symmetric algebras that are graded over an abelian group. This generalizes the main result obtained in [1] for cyclic groups.

## 2 Definitions and basic notions

### 2.1 $G$ -graded twisted algebras over fields.

**Definition 1.** Let  $G$  denote a group. A  $G$ -graded twisted algebra  $W$  is an algebra over a commutative ring  $R$  with a  $G$ -grading, i.e.,  $W = \bigoplus_{g \in G} W_g$ , with  $W_a W_b \subset W_{ab}$ . Each  $W_g$  is assumed to be a free  $R$ -module of rank one, and we demand that  $W$  is free of zero monomial divisors, i.e.,  $w_a \cdot w_b \neq 0$  for every non-zero elements  $w_a \in W_a$ ,  $w_b \in W_b$ . We also demand that  $W$  has an identity element  $1 \in W_e$ , where  $W_e$  denotes the graded component corresponding to the identity element  $e \in G$ . In this article  $R$  always will be a field.

As each graded component  $W_g$  is a vector space of dimension one, each choice of a non zero element  $w_g \in W_g$ , for each  $g \in G$ , produces a graded basis  $\mathcal{B} = \{w_g : g \in G\}$  for  $W$ . For each such basis there is a structure constant associated to it,  $C_{\mathcal{B}} : G \times G \rightarrow A$ , defined by the identity:  $w_a \cdot w_b = C_{\mathcal{B}}(a, b)w_{ab}$ . Here  $A \subset K^*$  must be a subgroup of the multiplicative group of all nonzero elements of  $K$ , since  $W$  has no zero divisor monomials. From now on we will omit the subscript  $\mathcal{B}$  if a particular basis is clear in the context.

Given a structure constant  $C : G \times G \rightarrow A$ , we define two functions  $q : G \times G \rightarrow A$  and  $r : G \times G \times G \rightarrow A$  as:

$$\begin{aligned} q(a, b) &= C(a, b)C(b, a)^{-1} \\ r(a, b, c) &= C(b, c)C(ab, c)^{-1}C(a, bc)C(a, b)^{-1} \end{aligned} \tag{1}$$

The associativity of elements of  $W$  can be described in terms of the function  $r : G \times G \times G \rightarrow A$  as follows:  $w_a \cdot (w_b \cdot w_c) = r(a, b, c)(w_a \cdot w_b) \cdot w_c$ . When  $G$  is an abelian group, the commutativity of elements of  $W$  is given in terms of the function  $q : G \times G \rightarrow A$  as  $w_a \cdot w_b = q(a, b)w_b \cdot w_a$ .

**Definition 2.** A morphism between two  $G$ -graded twisted  $K$ -algebras  $W = \bigoplus_{g \in G} W_g$  and  $V = \bigoplus_{g \in G} V_g$  is an unitarian homomorphism of  $K$ -algebras  $\varphi : W \rightarrow V$ . If the homomorphism preserves the grading, i.e.,  $\varphi(W_g) \subset V_g$ , we say the morphism is graded.

### 2.2 Group cohomology.

In this section we recall some basic definitions and results from the cohomology of groups that will be needed in the next section. We adhere to the terminology used in [10].

Let  $G$  be a group. A  $G$ -module  $M$  is an abelian group with an action which is compatible with the abelian group structure of  $M$ . Every abelian group  $M$  may be regarded as a  $G$ -module with the trivial action. Given a group  $G$  and  $A$ , an abelian group, the cohomology of  $G$  with coefficients in  $A$ , denoted by  $H^\bullet(G, A)$ , may be defined as  $\text{Ext}_{\mathbb{Z}[G]}^\bullet(\mathbb{Z}, A)$ . A very useful way to compute  $H^\bullet(G, A)$  is the following: Define  $C^n(G, A)$  as the set of functions from  $G^n =$  the product of  $n$  copies of  $G$ , to  $A$ , and consider  $\partial^n : C^n(G, A) \rightarrow C^{n+1}(G, A)$ , the boundary map given by the formula:

$$\partial^n(f)(g_1, \dots, g_{n+1}) = g_1 f(g_2, \dots, g_{n+1}) + \sum_{i=1}^n (-1)^i f(g_1, \dots, g_{i-1}, g_i g_{i+1}, \dots, g_{n+1}) + f(g_1, \dots, g_n).$$

Then, the cohomology  $H^\bullet(G, A)$  turns out to be the homology of the complex:

$$\dots \longrightarrow C^{n-1}(G, A) \xrightarrow{\partial^{n-1}} C^n(G, A) \xrightarrow{\partial^n} C^{n+1}(G, A) \longrightarrow \dots,$$

(see [10], page 19).

### 3 Computation of $H^2(G, A)$ when $G$ is a finite abelian group.

When  $G$  is a finite cyclic group there is a particular free resolution for  $\mathbb{Z}$ , regarded as a  $\mathbb{Z}[G]$ -modules with the trivial action. If we denote  $\mathbb{Z}[t]$  as the quotient  $\frac{\mathbb{Z}[T]}{(T^n-1)}$ , and let  $N$  be the sum  $1 + t + t^2 + \dots + t^{n-1}$ , then clearly  $\mathbb{Z} \cong \frac{\mathbb{Z}[t]}{(t-1)}$ , and the following complex is exact:

$$\dots \xrightarrow{N} \mathbb{Z}[t] \xrightarrow{t-1} \mathbb{Z}[t] \xrightarrow{N} \mathbb{Z}[t] \xrightarrow{t-1} \mathbb{Z}[t] \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0 \quad (2)$$

By suppressing  $\mathbb{Z}$ , and then applying the contravariant functor  $\text{Hom}_{\mathbb{Z}[t]}(-, A)$ , where  $A$  is regarded as a  $G$ -module with the trivial action, we obtain the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_{\mathbb{Z}[t]}(\mathbb{Z}[t], A) & \xrightarrow{t-1} & \text{Hom}_{\mathbb{Z}[t]}(\mathbb{Z}[t], A) & \xrightarrow{N} & \text{Hom}_{\mathbb{Z}[t]}(\mathbb{Z}[t], A) \xrightarrow{t-1} \text{Hom}_{\mathbb{Z}[t]}(\mathbb{Z}[t], A) \longrightarrow \dots \\ & & \lambda_0 \downarrow & & \lambda_1 \downarrow & & \lambda_2 \downarrow \\ 0 & \longrightarrow & A & \xrightarrow{D^0} & A & \xrightarrow{D^1} & A \xrightarrow{D^2} A \longrightarrow \dots \end{array} \quad (3)$$

The map  $\lambda_i : \text{Hom}_{\mathbb{Z}[t]}(\mathbb{Z}[t], A) \rightarrow A$  is the isomorphism given by  $\lambda_i(\varphi) = \varphi(1)$ , with inverse  $\lambda_i^{-1} : A \rightarrow \text{Hom}_{\mathbb{Z}[t]}(\mathbb{Z}[t], A)$  given by  $\lambda_i^{-1}(a) = \varphi_a$ , where  $\varphi_a(1) = a$  and  $D^j = \lambda_{j+1} \circ \delta_j \circ \lambda_j^{-1}$ . Here,  $\delta_0 = t - 1$ ,  $\delta_1 = N$ ,  $\delta_2 = t - 1$ , and  $\delta_3 = N$ . Thus,

$$D^1(a) = N\varphi_a(1) = N \cdot a = na, \text{ and } D^2(a) = (t - 1)\varphi_a(1) = (t - 1) \cdot a = 0.$$

Therefore, by taking the second homology we obtain  $H^2(G, A) \cong \ker(D^2)/\text{im}(D^1) = A/nA$ .

We will need the following lemma:

**Lemma 3.** *If  $F_\bullet \xrightarrow{\delta_\bullet} \mathbb{Z} \longrightarrow 0$  is a free resolution of  $\mathbb{Z}[G_1]$ -modules and  $F'_\bullet \xrightarrow{\delta'_\bullet} \mathbb{Z} \longrightarrow 0$  is a free resolution of  $\mathbb{Z}[G_2]$ -modules, then the tensor product of these two resolutions  $H_\bullet \xrightarrow{D_\bullet} \mathbb{Z} \rightarrow 0$  is a free resolution of  $\mathbb{Z}[G_1 \times G_2]$ -modules, where  $H_n = \bigoplus_{i=0}^n F_{n-i} \otimes F'_i$ , and for each  $a \otimes b \in F_{n-i} \otimes F'_i$ , the boundary homomorphisms are defined as*

$$D_n(a \otimes b) = \delta_{n-i}(a) \otimes b + (-1)^{n-i} a \otimes \delta'_i(b).$$

(See [10], page 107.)

Now we are ready to prove the main result of this section.

**Theorem 4.** *Let  $G$  be a finite abelian group, presented as  $G = G_1 \times \dots \times G_k$ , where  $G_i$  is a cyclic group of order  $n_i$ . Let  $A$  be any abelian group. Then*

$$H^2(G, A) \cong \left( \bigoplus_{i=0}^k \frac{A}{n_i A} \right) \oplus \left( \bigoplus_{1 \leq i < j \leq k} \text{Ann}_A(n_i) \cap \text{Ann}_A(n_j) \right).$$

*Proof.* By the lemma above, the tensor product of the free resolutions of  $\mathbb{Z}$ , regarded as a  $\mathbb{Z}[G_i]$ -module, gives a free resolution of  $\mathbb{Z}$  as a  $\mathbb{Z}[G]$ -module. On the other hand,  $\mathbb{Z}[G] \cong \mathbb{Z}[G_1] \times \dots \times \mathbb{Z}[G_k]$ . This product in turn is isomorphic to  $\mathbb{Z}[T_1, \dots, T_k]/(T_1^{n_1} - 1, \dots, T_k^{n_k} - 1) = \mathbb{Z}[t_1, \dots, t_k]$ , where  $t_i$  denotes the class of  $T_i$ .

We use induction on  $k$ : The previous lemma gives the following free resolution of  $\mathbb{Z}$  as a  $\mathbb{Z}[G_1 \times \cdots \times G_{k-1}]$ -module:

$$\begin{array}{ccccccc} & F_3 & & F_2 & & F_1 & & F_0 \\ & \parallel & & \parallel & & \parallel & & \parallel \\ \longrightarrow & \mathbb{Z}[t_1, \dots, t_{k-1}]^{a_{k-1}} & \xrightarrow{\Delta_2^{k-1}} & \mathbb{Z}[t_1, \dots, t_{k-1}]^{b_{k-1}} & \xrightarrow{\Delta_1^{k-1}} & \mathbb{Z}[t_1, \dots, t_{k-1}]^{c_{k-1}} & \xrightarrow{\Delta_0^{k-1}} & \mathbb{Z}[t_1, \dots, t_{k-1}] \longrightarrow \mathbb{Z} \longrightarrow 0 \end{array} \quad (4)$$

By tensoring with the following free resolution of  $\mathbb{Z}$  as a  $\mathbb{Z}[G_k]$ -module

$$\begin{array}{ccccccc} & F'_3 & & F'_2 & & F'_1 & & F'_0 \\ & \parallel & & \parallel & & \parallel & & \parallel \\ \longrightarrow & \mathbb{Z}[t_k] & \xrightarrow{\delta_2^k} & \mathbb{Z}[t_k] & \xrightarrow{\delta_1^k} & \mathbb{Z}[t_k] & \xrightarrow{\delta_0^k} & \mathbb{Z}[t_k] \longrightarrow \mathbb{Z} \longrightarrow 0, \end{array} \quad (5)$$

where  $\delta_0^k = t_k - 1$ , and  $\delta_1^k = N_k = 1 + t_k + \cdots + t_k^{n_k-1}$ , we obtain a free resolution of  $\mathbb{Z}$  as a  $\mathbb{Z}[G_1 \times \cdots \times G_k]$ -module:

$$\longrightarrow \mathbb{Z}[t_1, \dots, t_k]^{a_k} \xrightarrow{\Delta_2^k} \mathbb{Z}[t_1, \dots, t_k]^{b_k} \xrightarrow{\Delta_1^k} \mathbb{Z}[t_1, \dots, t_k]^{c_k} \xrightarrow{\Delta_0^k} \mathbb{Z}[t_1, \dots, t_k] \longrightarrow \mathbb{Z} \longrightarrow 0. \quad (6)$$

Here,

$$\mathbb{Z}[t_1, \dots, t_k]^{a_k} = \bigoplus_{i=0}^3 F_{3-i} \otimes F'_i = \mathbb{Z}[t_1, \dots, t_k]^{a_{k-1}} \oplus \mathbb{Z}[t_1, \dots, t_k]^{b_{k-1}} \oplus \mathbb{Z}[t_1, \dots, t_k]^{c_{k-1}} \oplus \mathbb{Z}[t_1, \dots, t_k],$$

$$\mathbb{Z}[t_1, \dots, t_k]^{b_k} = \bigoplus_{i=0}^2 F_{2-i} \otimes F'_i = \mathbb{Z}[t_1, \dots, t_k]^{b_{k-1}} \oplus \mathbb{Z}[t_1, \dots, t_k]^{c_{k-1}} \oplus \mathbb{Z}[t_1, \dots, t_k],$$

and

$$\mathbb{Z}[t_1, \dots, t_k]^{c_k} = \bigoplus_{i=0}^1 F_{1-i} \otimes F'_i = \mathbb{Z}[t_1, \dots, t_k]^{c_{k-1}} \oplus \mathbb{Z}[t_1, \dots, t_k].$$

Therefore,

$$\begin{aligned} a_k &= a_{k-1} + b_{k-1} + c_{k-1} + 1 = a_{k-1} + \binom{k}{2} + (k-1) + 1, \\ b_k &= b_{k-1} + c_{k-1} + 1 = \binom{k}{2} + (k-1) + 1 = \binom{k+1}{2}, \text{ and} \\ c_k &= c_{k-1} + 1 = (k-1) + 1 = k. \end{aligned}$$

Furthermore, we also get recursive equations for the boundary maps  $\Delta_0^k, \Delta_1^k$  and  $\Delta_2^k$ :

$$\begin{aligned} \Delta_0^k((p_1, \dots, p_{k-1}), q) &= \Delta_0^{k-1}(p_1, \dots, p_{k-1}) + \delta_0^k(q), \\ \Delta_1^k((p_1, \dots, p_{\binom{k}{2}}), (q_1, \dots, q_{k-1}), r) \\ &= (\Delta_1^{k-1}(p_1, \dots, p_{\binom{k}{2}}) - \delta_0^k(q_1, \dots, q_{k-1}), \Delta_0^{k-1}(q_1, \dots, q_{k-1}) + \delta_1^k(r)), \end{aligned}$$

$$\begin{aligned}
& \Delta_2^k((p_1, \dots, p_{a_{k-1}}), (q_1, \dots, q_{\binom{k}{2}}), (r_1, \dots, r_{k-1}), l) = \\
& = (\Delta_2^{k-1}(p_1, \dots, p_{a_{k-1}}) + \delta_0^k(q_1, \dots, q_{\binom{k}{2}}), \Delta_1^{k-1}(q_1, \dots, q_{\binom{k}{2}}) - \delta_1^k(r_1, \dots, r_{k-1}), \Delta_0^{k-1}(r_1, \dots, r_{k-1}) + \delta_2^k(l))
\end{aligned} \tag{7}$$

where  $\delta_i^k(a_1, \dots, a_n) = (\delta_i^k(a_1), \dots, \delta_i^k(a_n))$ . Suppressing  $\mathbb{Z}$ , and then applying the functor  $\text{Hom}_{\mathbb{Z}[G]}(-, A)$  in (6), we get a complex which is isomorphic to the complex:

$$0 \longrightarrow A \xrightarrow{D_k^0} A^{k-1} \oplus A \xrightarrow{D_k^1} A^{\binom{k}{2}} \oplus A^{k-1} \oplus A \xrightarrow{D_k^2} A^{a_{k-1}} \oplus A^{\binom{k}{2}} \oplus A^{k-1} \oplus A \longrightarrow \dots,$$

where the boundary maps  $D_k^1$  and  $D_k^2$  are also given recursively by:

$$\begin{aligned}
D_k^1((x_1, \dots, x_{k-1}), y) &= (D_{k-1}^1(x_1, \dots, x_{k-1}), 0, 0, \dots, 0, n_k \cdot y) \in A^{\binom{k}{2}} \oplus A^{k-1} \oplus A \\
&= (n_1 x_1, 0, n_2 x_2, 0, 0, n_3 x_3, 0, 0, 0, n_4 x_4, 0, 0, 0, 0, n_5 x_5, 0, 0, 0, 0, n_6 x_6, \dots, n_{k-1} x_{k-1}, 0, 0, \dots, 0, n_k y).
\end{aligned}$$

And by

$$\begin{aligned}
D_k^2((x_1, \dots, x_{b_{k-1}}), (y_1, \dots, y_{k-1}), z) &= \\
&= (D_{k-1}^2(x_1, \dots, x_{b_{k-1}}), D_{k-1}^1(y_1, \dots, y_{k-1}), -n_k y_1, \dots, -n_k y_{k-1}, 0) \in A^{a_{k-1}} \oplus A^{\binom{k}{2}} \oplus A^{k-1} \oplus A \\
&= (D_{k-1}^2(x_1, \dots, x_{b_{k-1}}), n_1 y_1, 0, n_2 y_2, 0, 0, n_3 y_3, 0, 0, 0, n_4 y_4, 0, 0, 0, 0, n_5 y_5, 0, 0, 0, 0, n_6 y_6, \dots, \\
&\dots, n_{k-2} y_{k-2}, 0, 0, \dots, 0, n_{k-1} y_{k-1}, -n_k y_1, \dots, -n_k y_{k-1}, 0).
\end{aligned}$$

Therefore,  $\text{im}(D_k^1) = \text{im}(D_{k-1}^1) \oplus 0 \oplus 0 \oplus \dots \oplus 0 \oplus n_k A$ , and

$$\ker(D_k^2) = \ker(D_{k-1}^2) \oplus \text{Ann}_A(n_1) \cap \text{Ann}_A(n_k) \oplus \dots \oplus \text{Ann}_A(n_{k-1}) \cap \text{Ann}_A(n_k) \oplus A.$$

Hence, we obtain the following recursive formula:

$$H^2(G, A) \cong \frac{\ker(D_{k-1}^2)}{\text{im}(D_{k-1}^1)} \oplus \text{Ann}_A(n_1) \cap \text{Ann}_A(n_k) \oplus \dots \oplus \text{Ann}_A(n_{k-1}) \cap \text{Ann}_A(n_k) \oplus \frac{A}{n_k A}.$$

By induction we know that

$$H^2(L, A) \cong \left( \bigoplus_{i=1}^{k-1} \frac{A}{n_i A} \right) \oplus \left( \bigoplus_{1 \leq i < j \leq k-1} \text{Ann}_A(n_i) \cap \text{Ann}_A(n_j) \right),$$

where  $L = G_1 \times \dots \times G_{k-1}$ . Since  $\ker(D_{k-1}^2)/\text{im}(D_{k-1}^1) = H^2(L, A)$ , it follows that

$$H^2(G, A) \cong \left( \bigoplus_{i=1}^k \frac{A}{n_i A} \right) \oplus \left( \bigoplus_{1 \leq i < j \leq k} \text{Ann}_A(n_i) \cap \text{Ann}_A(n_j) \right).$$

□

## 4 Classification of associative $G$ -graded twisted $K$ -algebras, when $G$ is a finite abelian group and $K = \mathbb{C}$ or $K = \mathbb{R}$ .

We start by recalling the following theorem that was proved in [1].

**Theorem 5.** *Let  $W = \oplus_{g \in G} W_g$  and  $V = \oplus_{g \in G} V_g$  be  $G$ -graded twisted  $K$ -algebras with fixed bases  $\mathcal{B}$  and  $\mathcal{B}'$  respectively, and let  $C_1, C_2$  the corresponding structure constants. Then  $W$  is graded-isomorphic to  $V$  if and only if the function  $C_1 C_2^{-1}$  is in the kernel of  $d^2 : C^2(G, K^*) \rightarrow C^3(G, K^*)$  and the class  $[C_1 C_2^{-1}]$  is trivial in  $H^2(G, K^*)$  ([1], Page 4).*

The above theorem implies that two associative  $G$ -graded twisted algebras  $W_1$  and  $W_2$  are isomorphic as graded algebras if and only if  $[C_1] = [C_2]$  in  $H^2(G, K^*)$ , where  $C_1, C_2 : G \times G \rightarrow K^*$  denote the structure constants for  $W_1$  and  $W_2$ , respectively, and where  $K^*$  is viewed as a trivial  $G$ -module. Hence, in this case the number of non-isomorphic associative  $G$ -graded twisted algebras is given by the cardinality of  $H^2(G, K^*)$ .

Assume  $G$  is a finite abelian group presented as  $G \cong Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k}$ . First, let us consider the case where  $K = \mathbb{C}$ .

As we showed before,

$$H^2(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}, \mathbb{C}^*) \cong \left( \bigoplus_{i=1}^k \frac{\mathbb{C}^*}{n_i \mathbb{C}^*} \right) \oplus \left( \bigoplus_{1 \leq i < j \leq k} \text{Ann}_{\mathbb{C}^*}(n_i) \cap \text{Ann}_{\mathbb{C}^*}(n_j) \right).$$

Note that  $\mathbb{C}^* = n \mathbb{C}^*$ , for all  $n \in \mathbb{N}$ . If  $d_{i,j} = \gcd(n_i, n_j)$  denotes the greatest common divisor for  $i, j = 1, 2, \dots, k$ , then  $\text{Ann}_{\mathbb{C}^*}(n_i) \cap \text{Ann}_{\mathbb{C}^*}(n_j) = \text{Ann}_{\mathbb{C}^*}(d_{i,j})$ . Thus, we have the following isomorphism:

$$H^2(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}, \mathbb{C}^*) \cong \bigoplus_{1 \leq i < j \leq k} \text{Ann}_{\mathbb{C}^*}(d_{i,j}) \cong \bigoplus_{1 \leq i < j \leq k} \mathbb{Z}_{d_{i,j}}.$$

Therefore, there are  $d$  non-isomorphic associative  $G$ -graded twisted  $\mathbb{C}$ -algebras, where  $d = \prod_{1 \leq i < j \leq k} d_{i,j}$ .

Now we deal with the case  $K = \mathbb{R}$ . We know that

$$H^2(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}, \mathbb{R}^*) \cong \left( \bigoplus_{i=1}^k \frac{\mathbb{R}^*}{n_i \mathbb{R}^*} \right) \oplus \left( \bigoplus_{1 \leq i < j \leq k} \text{Ann}_{\mathbb{R}^*}(n_i) \cap \text{Ann}_{\mathbb{R}^*}(n_j) \right).$$

By reorganizing the  $n_j$ 's we may assume, without loss of generality, that  $n_1, \dots, n_s$  are even and that  $n_{s+1}, \dots, n_k$  are odd. We know that  $\mathbb{R}^*/n\mathbb{R}^* = \{1\}$ , if  $n$  is odd, and  $\mathbb{R}^*/n\mathbb{R}^* = \{1, -1\}$ , if  $n$  is even, and that  $\text{Ann}_{\mathbb{R}^*}(n) = \{1\}$ , if  $n$  is odd, and  $\text{Ann}_{\mathbb{R}^*}(n) = \{1, -1\}$ , if  $n$  is even. Therefore,

$$H^2(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s} \times \mathbb{Z}_{n_{s+1}} \times \cdots \times \mathbb{Z}_{n_k}, \mathbb{R}^*) \cong \left( \bigoplus_{i=1}^s \mathbb{Z}_2 \right) \oplus \left( \bigoplus_{i=1}^{\frac{(s-1)s}{2}} \mathbb{Z}_2 \right) \cong \bigoplus_{i=1}^{\frac{s(s+1)}{2}} \mathbb{Z}_2.$$

This formula readily implies the following theorem.

**Theorem 6.** *Let  $G$  be written as a product  $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s} \times \mathbb{Z}_{n_{s+1}} \times \cdots \times \mathbb{Z}_{n_k}$  where  $n_1, \dots, n_s$  are even and  $n_{s+1}, \dots, n_k$  are odd. Then there are  $2^{s(s+1)/2}$  non-isomorphic associative  $G$ -graded twisted  $\mathbb{R}$ -algebras.*

In the next section we deal with the classification problem in the non-associative case. We study those algebras satisfying a type of symmetry that will be called  $(1, 2)$ -symmetry.

## 5 Classification of $(1, 2)$ -symmetric $G$ -graded twisted $\mathbb{C}$ -algebras.

We recall that given  $W$  a  $G$ -graded twisted  $K$ -algebra with a fixed basis  $\mathcal{B}$  and structure constant  $C : G \times G \rightarrow K^*$ , the associative function  $r : G \times G \times G \rightarrow K^*$  was defined as  $r(a, b, c) = C(b, c)C(ab, c)^{-1}C(a, bc)C(a, b)^{-1}$ .

**Remark 7.** *The function  $r : G \times G \times G \rightarrow K^*$  does not depend on the choice basis of  $W$  (see [1]).*

**Definition 8.** *Let  $W$  be a  $G$ -graded twisted  $K$ -algebra. We say that  $W$  is  $(1, 2)$ -symmetric if  $r(a, b, c) = r(b, a, c)$  for every  $a, b, c \in G$ .*

In [1], it was proved that for  $G \cong \mathbb{Z}_n$ , a cyclic group, the number of non-(graded) isomorphic  $(1, 2)$ -symmetric  $G$ -graded twisted  $\mathbb{C}$ -algebras with structure constants taking values in a finite subgroup  $A \subset \mathbb{C}^*$  is given by  $|R_n|^{|G|-2} = |R_n|^{n-2}$ , where  $R_n$  denotes the set of  $n$ -th roots of unity in  $A$ . In this section, we provide a generalization of the arguments used in [1] that will allow us to state an equivalent result for the case of any finite abelian group. For the sake of clarity we will mainly focus on groups that are the product of only two cyclic groups. At the end of this section we deal with finite abelian groups in general. Since the arguments are almost identical to the case of groups that are the product of two factors, we will limit ourselves to sketch the main arguments.

Suppose  $G$  is presented as a product  $\mathbb{Z}_m \times \mathbb{Z}_n$ , and consider  $W = \bigoplus_{a,b \in G} W_{a,b}$  a  $(1, 2)$ -symmetric  $G$ -graded twisted  $\mathbb{C}$ -algebra, with a fixed basis  $\{x_{a,b} : x_{a,b} \in W_{a,b}\}$  and structure constant  $\tilde{C} : G \times G \rightarrow A \subset \mathbb{C}^*$ ,  $A$  a finite subgroup of  $\mathbb{C}^*$ . When  $G = \mathbb{Z}_n$  is a cyclic group with generator  $g \in G$ , a basis for  $W$  was called *standard* if it had the form  $\{1, w_g^{(1)}, w_g^{(2)}, \dots, w_g^{(n-1)}\}$  where  $w_g^{(i)} = w_g \cdot w_g^{(i-1)}$  and  $w_g \cdot w_g^{(n-1)} = 1$  [1]. We generalize this construction as follows. We may think of the graduation of  $W$  as an array of the following form:

$$\begin{array}{ccccccccc}
 W_{0,0} & & W_{0,1} & & W_{0,2} & & \cdots & & W_{0,n-1} & \cdot \\
 \\
 W_{1,0} & & W_{1,1} & & W_{1,2} & & \cdots & & W_{1,n-1} & \\
 \\
 \vdots & & \vdots & & \vdots & & \cdots & & \vdots & \\
 \\
 W_{m-1,0} & & W_{m-1,1} & & W_{m-1,2} & & \cdots & & W_{m-1,n-1} & 
 \end{array}$$

As in the cyclic case, we choose standard bases for the first row and first column. These two bases will be denoted by  $\{1, w_{0,1}, w_{0,2}, \dots, w_{0,n-1}\}$  and  $\{1, w_{1,0}, w_{2,0}, \dots, w_{m-1,0}\}$ , respectively. Now for the  $i$ -th row define  $w_{i,j} = w_{0,1} \cdot w_{i,j-1}$  for  $j = 1, \dots, n-1$ . We notice that  $w_{0,1} \cdot w_{i,n-1} \in W_{i,0}$ . Hence,  $w_{0,1} \cdot w_{i,n-1} = \alpha_i \cdot w_{i,0}$ , for some  $\alpha_i \in A$ . We call  $\mathcal{B} = \{w_{i,j} : i = 0, \dots, m-1, j = 0, \dots, n-1\}$  a *standard basis* for  $W$ .

Define  $T_{i,j} : W \rightarrow W$  to be the linear transformation given by  $T_{i,j}(x) = w_{i,j} \cdot x$ . For  $T_{0,1}$ , its

rational form consists of  $m$ -blocks where each one looks like

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \alpha_i \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

where  $\alpha_0 = 1, \alpha_1, \dots, \alpha_{m-1}$  are elements in  $A$  such that  $T_{0,1}(w_{i,n-1}) = \alpha_i \cdot w_{i,0}$ . Let us denote by  $\{e_{i,j}\}_{j=0,\dots,n-1}$  the  $n$ -th roots of  $\alpha_i$ , for  $i = 0, \dots, m-1$ . A straightforward computation shows that  $e_{i,j}$  is an eigenvalue of  $T_{0,1}$  with eigenvector

$$z_{i,j} = \sum_{k=0}^{n-1} e_{i,j}^{-k} \cdot w_{i,k}. \quad (8)$$

From now on, the elements of the group  $G$  will be denoted by products of the form  $a^r b^s$ , with  $0 \leq r \leq m-1$ ,  $0 \leq s \leq n-1$ . Also we will write  $w_{r,s} \cdot w_{i,j} = C(a^r b^s, a^i b^j) \cdot w_{r+i, s+j}$ . Notice that  $T_{r,s} \circ T_{i,j}(x) = q(a^r b^s, a^i b^j) \cdot T_{i,j} \circ T_{r,s}(x)$  for every  $x \in W$ . Here,  $q(a^r b^s, a^i b^j) = C(a^r b^s, a^i b^j) C(a^i b^j, a^r b^s)^{-1}$  (see [1]). Therefore,

$$\begin{aligned} T_{0,1}(T_{r,s}(z_{i,j})) &= q(b, a^r b^s) T_{r,s}(T_{0,1}(z_{i,j})) = q(b, a^r b^s) T_{r,s}(e_{i,j} z_{i,j}) \\ &= q(b, a^r b^s) e_{i,j} T_{r,s}(z_{i,j}). \end{aligned}$$

Hence,  $T_{r,s}(z_{i,j})$  is an eigenvector of  $T_{0,1}$  associated to the eigenvalue  $q(b, a^r b^s) e_{i,j}$ . Since

$$T_{r,s}(z_{i,j}) = w_{r,s} \cdot \sum_{k=0}^{n-1} e_{i,j}^{-k} w_{i,k} = \sum_{k=0}^{n-1} e_{i,j}^{-k} C(a^r b^s, a^i b^k) w_{[r+i], [s+k]}, \quad (9)$$

where  $[\ ]$  denotes the equivalence class in  $\mathbb{Z}_n$  or  $\mathbb{Z}_m$ , we deduce that

$$T_{r,s}(z_{i,j}) = \eta_{r,s}^{i,j} \cdot z_{[r+i], l}, \quad (10)$$

for some  $l \in \{0, 1, \dots, n-1\}$  and some  $\eta_{r,s}^{i,j} \in K^*$ . Also, since  $q(b, a^r b^s) e_{i,j}$  is an eigenvalue associated to  $T_{r,s}(z_{i,j})$  we see that

$$q(b, a^r b^s) e_{i,j} = e_{[r+i], l}. \quad (11)$$

By definition  $e_{i,j}^n = \alpha_i$ . Therefore,

$$q(b, a^r b^s)^n \alpha_i = \alpha_{[r+i]}. \quad (12)$$

It follows from the definition of a standard basis that  $C(b, a^r b^s) = 1$ , if  $s \neq n-1$ , and that  $C(b, a^r b^{n-1}) = \alpha_r$ . Therefore, from equation (12) we obtain  $\alpha_r = C(a^r b^s, b)^{-n}$ ,  $r = 1, 2, \dots, m$ , when  $s \neq n-1$ , and  $\alpha_r^{n-1} = C(a^r b^{n-1}, b)^n$ ,  $r = 1, 2, \dots, m$ . Hence, as  $q(b, a^r b^s) = C(a^r b^s, b)^{-1}$ , when  $s \neq n-1$ , by replacing  $q(b, a^r b^s)$  in equation (12), we get  $\alpha_r \alpha_i = \alpha_{[r+i]}$ . From this we see that  $\alpha_i = \alpha_1^i$ . Finally, notice that  $\alpha_i = C(a, b)^{-in}$ . Since  $\alpha_1^m = 1$ , then  $C(a, b)^{mn} = 1$ .



We summarize below what we have obtained so far:

$$\begin{aligned}
\alpha_i &= \alpha_1^i, \\
\alpha_i &= C(a, b)^{-in}, \\
C(a^r b^s, b)^{-n} &= \alpha_r = (C(a, b)^{-r})^n, \quad \text{for } s \neq n-1, \\
C(a^r b^{n-1}, b)^n &= \alpha_r^{n-1} = (C(a, b)^{-r(n-1)})^n, \\
C(b^s, b)^n &= 1, \quad (\text{cyclic case}) \\
C(a^r, a)^m &= 1, \quad (\text{cyclic case}) \\
C(a, b)^{mn} &= 1.
\end{aligned} \tag{13}$$

On the other hand, by equation (11) it follows that

$$\eta_{r,s}^{i,j} \cdot Z_{[r+i],l} = \sum_{k=0}^{n-1} \eta_{r,s}^{i,j} \cdot e_{[r+i],l}^{-k} \cdot w_{[r+i],k} = \sum_{k=0}^{n-1} \eta_{r,s}^{i,j} \cdot q(b, a^r b^s)^{-k} \cdot e_{i,j}^{-k} \cdot w_{[r+i],k}.$$

Equations (9) and (10) imply the following identity:

$$\sum_{k=0}^{n-1} e_{i,j}^{-k} C(a^r b^s, a^i b^k) w_{[r+i],[s+k]} = \sum_{k=0}^{n-1} \eta_{r,s}^{i,j} \cdot q(b, a^r b^s)^{-k} \cdot e_{i,j}^{-k} \cdot w_{[r+i],k}.$$

But the last equation is equivalent to

$$\sum_{k=s}^{n-1} e_{i,j}^{-(k-s)} C(a^r b^s, a^i b^{k-s}) w_{[r+i],k} = \sum_{k=s}^{n-1} \eta_{r,s}^{i,j} \cdot q(b, a^r b^s)^{-k} \cdot e_{i,j}^{-k} \cdot w_{[r+i],k},$$

and also equivalent to

$$\sum_{k=0}^{s-1} e_{i,j}^{-(n+k-s)} C(a^r b^s, a^i b^{n+k-s}) w_{[r+i],k} = \sum_{k=0}^{s-1} \eta_{r,s}^{i,j} \cdot q(b, a^r b^s)^{-k} \cdot e_{i,j}^{-k} \cdot w_{[r+i],k} \quad \text{for } s \neq 0.$$

Therefore, the equations

$$e_{i,j}^{-(k-s)} \cdot C(a^r b^s, a^i b^{k-s}) = \eta_{r,s}^{i,j} \cdot q(b, a^r b^s)^{-k} \cdot e_{i,j}^{-k} \quad \text{for } k = s, s+1, \dots, n-1, \tag{14}$$

and

$$e_{i,j}^{-(n+k-s)} \cdot C(a^r b^s, a^i b^{n+k-s}) = \eta_{r,s}^{i,j} \cdot q(b, a^r b^s)^{-k} \cdot e_{i,j}^{-k} \quad \text{for } k = 0, 1, \dots, s-1, \quad \text{for } s \neq 0 \tag{15}$$

hold.

From (14) if we let  $k = s$ , we deduce that  $\eta_{r,s}^{i,j} = C(a^r b^s, a^i) \cdot q(b, a^r b^s)^s \cdot e_{i,j}^s$ . Replacing the last equation in (14) and in (15) we get that:

$$C(a^r b^s, a^i b^l) = C(a^r b^s, a^i) \cdot C(b, a^r b^s)^{-l} \cdot C(a^r b^s, b)^l, \quad \text{if } 0 \leq l < n-s, \tag{16}$$

and

$$C(a^r b^s, a^i b^l) = C(a^r b^s, a^i) \cdot C(b, a^r b^s)^{n-l} \cdot C(a^r b^s, b)^{l-n} \cdot \alpha_i, \quad \text{if } n-s \leq l \leq n-1; \quad s \neq 0. \tag{17}$$

Now we take in account the symmetry condition of  $r$ :  $r(a, b, c) = r(b, a, c)$  for every  $a, b, c \in G$ . For any three general elements  $a^r b^s, a^i b^k, a^j b^l \in G$  the symmetry condition looks like:

$$C(a^i b^k, a^j b^l) C(a^r b^s, a^{i+j} b^{k+l}) C(a^r b^s, a^i b^k)^{-1} = C(a^r b^s, a^j b^l) C(a^i b^k, a^{r+j} b^{s+l}) C(a^i b^k, a^r b^s)^{-1}. \quad (18)$$

Taking  $k = 0$  and  $l = 0$  in the above equation, and using the equation (16) and the fact that  $C(a^i, a^j) = C(a^i, a)^j$  (cyclic case, see [1]) we obtain:  $C(a^r b^s, a^{i+j}) = C(a^r b^s, a^i) C(a^r b^s, a^j)$ . Then, recursively, we get  $C(a^r b^s, a^j) = C(a^r b^s, a)^j$ , and therefore  $C(a^r b^s, a)^m = 1$ . Also, notice that since  $C(a^r b^s, b)^{-n} = \alpha_r$  when  $s \neq n - 1$  and  $C(a^r b^{n-1}, b)^{-n} = \alpha_r^{1-n}$ , we can rewrite equations (16) and (17) in the following manner:

$$\begin{aligned} C(a^r b^s, a^i b^l) &= C(a^r b^s, a)^i C(a^r b^s, b)^l, \quad \text{for } 0 \leq l \leq n - s - 1, \\ C(a^r b^s, a^i b^l) &= C(a^r b^s, a)^i C(a^r b^s, b)^l \alpha_r \alpha_i, \quad \text{for } n - s \leq l \leq n - 1 \quad \text{and } s \neq n - 1, \\ C(a^r b^{n-1}, a^i b^l) &= C(a^r b^s, a)^i C(a^r b^s, b)^l \alpha_r^{1-l} \alpha_i. \end{aligned}$$

We conclude that for a  $(1, 2)$ -symmetric  $\mathbb{Z}_m \times \mathbb{Z}_n$ -graded twisted  $\mathbb{C}$ -algebra the structure constant  $C : G \times G \rightarrow A$  referred to a standard basis  $\mathcal{B}$  must satisfy the following equations:

$$\begin{aligned} C(a^r b^s, a^i b^l) &= C(a^r b^s, a)^i C(a^r b^s, b)^l, \quad \text{for } 0 \leq l \leq n - s - 1, \\ C(a^r b^s, a^i b^l) &= C(a^r b^s, a)^i C(a^r b^s, b)^l \alpha_r \alpha_i, \quad \text{for } n - s \leq l \leq n - 1 \quad \text{and } s \neq n - 1, \quad s \neq 0, \\ C(a^r b^{n-1}, a^i b^l) &= C(a^r b^{n-1}, a)^i C(a^r b^{n-1}, b)^l \alpha_r^{1-l} \alpha_i \quad \text{if } 1 \leq l \leq n - 1, \\ C(a^r b^s, a)^m &= 1, \\ \alpha_i &= \alpha_1^i, \\ \alpha_i &= C(a, b)^{-in}, \\ C(a^r b^s, b)^{-n} &= \alpha_r = (C(a, b)^{-r})^n, \quad \text{for } s \neq n - 1, \\ C(a^r b^{n-1}, b)^n &= \alpha_r^{n-1} = (C(a, b)^{-r(n-1)})^n, \\ C(b^s, b)^n &= 1, \quad (\text{cyclic case}) \\ C(a^r, a)^m &= 1, \quad (\text{cyclic case}) \\ C(a, b)^{mn} &= 1. \end{aligned} \quad (19)$$

Now we prove that two  $(1, 2)$ -symmetric  $G$ -graded twisted algebras  $W_1$  and  $W_2$  are graded-isomorphic if and only if their structure constants referred to standard bases are the same.

**Theorem 9.** *Let  $W_1$  and  $W_2$  be  $(1, 2)$ -symmetric  $G$ -graded twisted  $\mathbb{C}$ -algebras with standard bases  $\mathcal{B}_1$  and  $\mathcal{B}_2$ , respectively, and associated structure constants  $C_1, C_2 : G \times G \rightarrow A$ . Then,  $W_1$  is graded-isomorphic to  $W_2$  if and only if  $C_1 = C_2$ .*

*Proof.* Suppose that  $W_1$  is graded-isomorphic to  $W_2$ . By Theorem 5,  $r_1 = r_2$  and  $[C_1] = [C_2]$  in  $H^2(G, A)$ . If  $[C_1] = [C_2]$ , then there exists  $\rho : G \rightarrow A$  such that  $C_1 = \partial^1(\rho) C_2$ . That is:

$$C_1(a^r b^s, a^i b^k) = \rho(a^i b^k) \rho(a^{r+i} b^{s+k})^{-1} \rho(a^r b^s) C_2(a^r b^s, a^i b^k). \quad (20)$$

Since the structure constants  $C_1, C_2$  are referred to standard bases, the following equalities hold for  $j = 1, 2$ :

$$\begin{aligned} C_j(1, a^i b^k) &= C_j(a^i b^k, 1) = 1, \quad \text{for all } i, k. \\ C_j(a, a^i) &= 1, \quad \text{for } i = 0, 1, \dots, m - 1. \\ C_j(b, a^i b^k) &= 1, \quad \text{for } i = 0, 1, \dots, m - 1, \quad k = 0, 1, \dots, n - 2. \\ C_j(b, a^i b^{n-1}) &= \alpha_{i, (j)}. \end{aligned}$$

These identities together with equation (20) yield:  $\rho(a^i) = \rho(a)^i$  for all  $i = 0, 1, 2, \dots, n-1$ . In particular,  $\rho(1) = 1$  and  $\rho(a)^m = 1$ , and

$$\rho(a^i b^k) = \rho(a)^i \rho(b)^k \text{ for } k \neq n.$$

Moreover,  $\rho(b^i) = \rho(b)^i$  for  $i = 0, 1, \dots, n$ , since

$$\begin{aligned} 1 &= C_1(b, b^i) = \rho(b^i) \rho(b^{i+1})^{-1} \rho(b) C_2(b, b^i) \\ &= \rho(b^i) \rho(b^{i+1})^{-1} \rho(b). \end{aligned}$$

Therefore,  $\rho(b)^n = \rho(b^n) = 1$ . All this can be summarize by saying that  $\rho : G \rightarrow A$  is a group homomorphism. It immediately follows  $\partial^1(\rho) \equiv 1$  what implies that  $C_1 = C_2$ . The reciprocal is clear.  $\square$

Finally, we want to see that if  $C : G \times G \rightarrow A$  is a function satisfying the identities stated in (19) then the vector space  $\mathbb{C}^m \times \mathbb{C}^n$ , endowed with the structure of a  $G$ -graded twisted algebra defined by the functions  $C$  (referred to the canonical basis of  $\mathbb{C}^m \times \mathbb{C}^n$ ) is a  $(1, 2)$ -symmetric  $G$ -graded twisted  $\mathbb{C}$ -algebra.

**Theorem 10.** *Let  $G = \mathbb{Z}_m \times \mathbb{Z}_n$  and let  $A \subset \mathbb{C}^*$  be a finite subgroup. Suppose that we choose values in  $A$  for  $C(a^r b^s, a)$  and  $C(a^r b^s, b)$  satisfying the identities in (19). Then  $W = \mathbb{C}^m \times \mathbb{C}^n$  with the multiplication given by  $C$  (referred to the canonical basis of  $\mathbb{C}^m \times \mathbb{C}^n$ ) is a  $(1, 2)$ -symmetric  $G$ -graded twisted  $\mathbb{C}$ -algebra.*

*Proof.* For  $0 \leq r, i \leq m-1$ , and  $0 \leq s, l \leq n-1$ , we define

$$f(r, s, i, l) = \begin{cases} 1 & \text{if } 0 \leq l \leq n-s-1 \\ \alpha_r \alpha_i & \text{if } n-s \leq l \leq n-1, \text{ and } s \neq n-1 \\ \alpha_r^{1-l} \alpha_i & \text{if } s = n-1, 1 \leq l \leq n-1 \end{cases}$$

From equation (19) we deduce the identity

$$C(a^r b^s, a^i b^l) = C(a^r b^s, a)^i C(a^r b^s, b)^l f(r, s, i, l).$$

But we know that  $r(a^r b^s, a^i b^k, a^j b^l) = r(a^i b^k, a^r b^s, a^j b^l)$  if and only if

$$C(a^i b^k, a^j b^l) C(a^r b^s, a^{i+j} b^{k+l}) C(a^r b^s, a^i b^k)^{-1} = C(a^r b^s, a^j b^l) C(a^i b^k, a^{r+j} b^{s+l}) C(a^i b^k, a^r b^s)^{-1}.$$

Therefore, this equation holds if and only if

$$\begin{aligned} f(i, k, j, l) f(r, s, i+j, [k+l]) f(r, s, i, k)^{-1} C(a^r b^s, b)^{[k+l]-(k+l)} = \\ = f(r, s, j, l) f(i, k, r+j, [s+l]) f(i, k, r, s)^{-1} C(a^i b^k, b)^{[s+l]-(s+l)}, \end{aligned} \quad (21)$$

where  $[ ]$  denotes residue classes in  $\mathbb{Z}_n$ . It is not difficult to see that equation (21) holds by directly computing from the identities in (19). For this, each one of the following cases should be considered separately:

$$\begin{aligned} [k+l] + s < n \quad \text{and} \quad [s+l] + k < n : & \begin{cases} k+l \geq n, s+l \geq n \\ k+l < n, s+l < n \end{cases} \\ [k+l] + s < n \quad \text{and} \quad [s+l] + k \geq n : & \begin{cases} k+l \geq n, s+l < n \end{cases} \\ [k+l] + s \geq n \quad \text{and} \quad [s+l] + k < n : & \begin{cases} s+l \geq n, k+l < n \end{cases} \\ [k+l] + s \geq n \quad \text{and} \quad [s+l] + k \geq n : & \begin{cases} k+l \geq n, s+l \geq n \\ k+l < n, s+l < n \end{cases} \end{aligned}$$

$\square$

We are ready to state the main theorem of this section:

**Theorem 11.** *The number of (graded) isomorphism classes of  $(1, 2)$ -symmetric  $\mathbb{Z}_m \times \mathbb{Z}_n$ -graded twisted  $\mathbb{C}$ -algebras with structure constants taking values in a finite subgroup  $A \subset \mathbb{C}^*$  is given by:*

$$|R_m|^{mn-3}|R_n|^{mn-3}|R_{mn}|,$$

where  $R_k$  denotes the set of  $k$ -th roots of unity:  $\{\omega \in A : \omega^k = 1\}$ .

*Proof.* From the discussion above, a  $(1, 2)$ -symmetric  $\mathbb{Z}_m \times \mathbb{Z}_n$ -graded twisted  $\mathbb{C}$ -algebra is determined, up to graded isomorphisms, by the structure constant that is defined with respect to a standard basis. In turn, this function is completely determined by all possible choices of  $C(a^r b^s, a)$  and  $C(a^r b^s, b)$ , satisfying the identities in (19). As  $C(a^r b^s, a)^m = 1$  for all  $0 \leq r \leq m-1$ ,  $0 \leq s \leq n-1$ , and

$$1 = C(1, a) = C(a, a) = C(b, a),$$

then we see that there are  $|R_m|^{mn-3}$  possible choices for  $C(a^r b^s, a)$ . Similarly, as  $C(b^s, b)^n = 1$  for  $0 \leq s \leq n-1$ , and  $C(1, b) = 1 = C(b, b)$ , then  $C(b^s, b)$  may be chosen in  $|R_n|^{n-2}$  possible ways. Since  $C(a, b)^{mn} = 1$ , there are  $|R_{mn}|$  possible values for  $C(a, b)$ . In the case where  $s \neq n-1$  and  $r \neq 0$ , the identities in (19) tell us that

$$C(a^r b^s, b)^n = C(a, b)^{rn} = (C(a, b)^r)^n.$$

Therefore,  $C(a^r b^s, b) = \omega C(a, b)^r$ , where  $\omega$  is some fixed  $n$ -th root of unity. Thus, there are  $|R_n|^{(n-1)(m-1)-1}$  possible choices for  $C(a^r b^s, b)$ , if  $s \neq n-1$  and  $r \neq 0$ .

Finally, again by using (19) we obtain:  $C(a^r b^{n-1}, b)^n = (C(a, b)^{-r(n-1)})^n$ , and therefore  $C(a^r b^{n-1}, b) = \omega C(a, b)^{-r(n-1)}$ , where  $\omega^n = 1$ . Hence, if  $r \geq 1$ , the value of  $C(a^r b^{n-1}, b)$  can be chosen in  $|R_n|^{m-1}$  possible manners. In conclusion, the number of algebras satisfying the hypothesis of the theorem is given by

$$|R_m|^{mn-3}|R_n|^{n-2+(n-1)(m-1)-1+m-1}|R_{mn}| = |R_m|^{mn-3}|R_n|^{mn-3}|R_{mn}|.$$

□

**Remark 12.** *If  $m$  and  $n$  are relatively prime, then  $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ , and since  $|R_{mn}| = |R_m||R_n|$ , the above number is equal to  $|R_{mn}|^{mn-2}$  which gives the correct number of non-isomorphic algebras in the cyclic case, as provided in [1].*

Now, we discuss the general case of any finite abelian group, presented as  $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ . Since the constructions are almost the same as the case of the product of two cyclic groups, in the rest of this article we will limit ourselves to give a sketch of the main arguments.

First, we start by defining a generalized standard basis for a  $G$ -graded twisted algebra  $W$ . We proceed by induction on the number of factors. Suppose  $G = G_1 \times \mathbb{Z}_{n_k}$ , where  $G_1 = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{k-1}}$ . Fix  $a_i \in \mathbb{Z}_{n_i}$  a generator. We define a standard basis for  $W$  as a basis of the form  $\{w_{g,j}\}$ , where  $w_{e,j} = w_{a_k}^{(j)}$  ( $e$  denotes the identity element of  $G_1$ ), as it was defined in the cyclic case, and where  $\{w_{g,0} = w_g : g \in G_1\}$  is a standard basis for  $W$  restricted to  $G_1$ , and  $w_{g,j} = w_{a_k}^{(1)} \cdot w_{g,j-1}$ , for  $g \neq e$  and  $j \neq 0$ . So for each  $g \in G_1$  there is  $\alpha_g \in A$  such that  $w_{a_k}^{(1)} \cdot w_{g,n_k-1} = \alpha_g \cdot w_g$ .

**Remark 13.** *Notice that since  $\mathcal{B}$  was defined recursively, its restriction to  $H = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_t}$  with  $1 \leq t \leq k$ , is a standard basis for  $W|_H$ .*

Let  $W$  be a  $(1, 2)$ -symmetric  $G$ -graded twisted  $\mathbb{C}$ -algebra, with standard basis  $\mathcal{B}$  and structure constant  $C : G \times G \rightarrow A$ . Consider as before  $T_{a_k} : W \rightarrow W$  the linear transformation given by  $T_{a_k}(x) = w_{a_k} \cdot x$ . For each  $g \in G_1$ , let  $\{e_{g,j}\}$  denotes the set of  $n_k$ -th roots of  $\alpha_g$ . Notice that  $\alpha_e = 1$  and therefore  $\{e_{e,j}\}$  is the set of  $n_k$ -roots of unity. Define

$$z_{g,j} = \sum_{\mu=0}^{n_k-1} e_{g,j}^{-\mu} w_{g,\mu}. \quad (22)$$

Then, as before,  $z_{g,j}$  is an eigenvector of  $T_{a_k}$  associated to the eigenvalue  $e_{g,j}$ . Since  $W$  is  $(1, 2)$ -symmetric we have

$$T_{a_k}(T_{g',s}(z_{g,j})) = q(a_k, g' \cdot a_k^s) \cdot e_{g,j} \cdot T_{g',s}(z_{g,j}),$$

(see [1]). Therefore  $T_{g',s}(z_{g,j})$  is an eigenvector of  $T_{a_k}$  associated to the eigenvalue  $q(a_k, g' \cdot a_k^s) \cdot e_{g,j}$ . Here,  $T_{g',s}$  denotes the linear transformation given by  $T_{g',s}(x) = w_{g',s} \cdot x$ . Since

$$T_{g',s}(z_{g,j}) = \sum_{\mu=0}^{n_k-1} e_{g,j}^{-\mu} \cdot C(g' \cdot a_k^s, g \cdot a_k^\mu) \cdot w_{g',s,[s+\mu]} \quad (23)$$

where  $[ ]$  denotes residue classes in  $\mathbb{Z}_{n_k}$ , then it holds that

$$T_{g',s}(z_{g,j}) = \eta_{g',s}^{g,j} \cdot z_{g',l}, \quad \text{for some } 0 \leq l \leq n_k - 1. \quad (24)$$

Hence,

$$q(a_k, g' \cdot a_k^s) \cdot e_{g,j} = e_{g',l}. \quad (25)$$

This last equation is the generalization of equation (11). As in that case, we may derive the following identities: If  $g = a_1^{r_1} a_2^{r_2} \dots a_{k-1}^{r_{k-1}}$ , then

$$\begin{aligned} \alpha_g &= C(a_1, a_k)^{-r_1 n_k} \dots C(a_{k-1}, a_k)^{-r_{k-1} n_k} \\ \alpha_{a_i} &= C(a_i, a_k)^{-n_k} \\ C(a_k, g \cdot a_k^{n_k-1}) &= \alpha_g = C(g, a_k)^{-n_k} \\ C(g \cdot a_k^s, a_k) &= \omega_{g,s} \cdot C(a_1, a_k)^{r_1} \dots C(a_{k-1}, a_k)^{r_{k-1}}, \quad \text{for } s \neq n_k - 1, \quad \text{where } \omega_{g,s}^{n_k} = 1 \\ C(g \cdot a_k^{n_k-1}, a_k) &= \omega_g \cdot C(a_1, a_k)^{-r_1(n_k-1)} \dots C(a_{k-1}, a_k)^{-r_{k-1}(n_k-1)} \quad \text{where } \omega_g^{n_k} = 1 \\ C(a_i, a_k)^{n_i n_k} &= 1. \end{aligned} \quad (26)$$

Now, equations (23), (24) and (25) imply:

$$\begin{aligned} \sum_{\mu=0}^{n_k-1} e_{g,j}^{-\mu} \cdot C(g' \cdot a_k^s, g \cdot a_k^\mu) \cdot w_{g',s,[s+\mu]} &= \eta_{g',s}^{g,j} \cdot z_{g',l} \\ &= \sum_{\mu=0}^{n_k-1} \eta_{g',s}^{g,j} \cdot e_{g',l}^{-\mu} \cdot w_{g',g,\mu} \\ &= \sum_{\mu=0}^{n_k-1} \eta_{g',s}^{g,j} \cdot q(a_k, g' \cdot a_k^s)^{-\mu} \cdot e_{g,j}^{-\mu} \cdot w_{g',g,\mu} \end{aligned}$$

Similarly, as in the case of a product of two cyclic groups, the last equation implies that for every  $g, g' \in G_1$ ,

$$\begin{aligned} C(g' \cdot a_k^s, g \cdot a_k^l) &= C(g' \cdot a_k^s, g) \cdot C(g' \cdot a_k^s, a_k)^l, \quad \text{if } 0 \leq l < n_k - s. \\ C(g' \cdot a_k^s, g \cdot a_k^l) &= C(g' \cdot a_k^s, g) \cdot C(g' \cdot a_k^s, a_k)^{l-n_k} \cdot \alpha_g, \quad \text{if } n_k - s \leq l \leq n_k - 1; \quad s \neq 0, \quad s \neq n_k - 1. \\ C(g' \cdot a_k^{n_k-1}, g \cdot a_k^l) &= C(g' \cdot a_k^{n_k-1}, g) \cdot C(g' \cdot a_k^{n_k-1}, a_k)^{l-n_k} \cdot \alpha_{g'}^{n_k-l} \cdot \alpha_g. \end{aligned} \quad (27)$$

That proves the following theorem:

**Theorem 14.** *Suppose that  $G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ , and fix generators  $a_1, a_2, \dots, a_k, a_i \in \mathbb{Z}_{n_i}$ . Suppose that  $W$  is a  $(1, 2)$ -symmetric  $G$ -graded twisted  $\mathbb{C}$ -algebra. If  $\mathcal{B}$  is a standard basis for  $W$  with structure constant  $C : G \times G \rightarrow \mathbb{C}^*$ , then:*

*If  $0 \leq l < n_k - s$  :*

$$C(g' \cdot a_k^s, g \cdot a_k^l) = C(g' \cdot a_k^s, g) \cdot C(g' \cdot a_k^s, a_k)^l,$$

*If  $n_k - s \leq l \leq n_k - 1, \quad s \neq 0, \quad s \neq n_k - 1$  :*

$$C(g' \cdot a_k^s, g \cdot a_k^l) = C(g' \cdot a_k^s, g) \cdot C(g' \cdot a_k^s, a_k)^{l-n_k} \cdot \alpha_g.$$

*For  $s = n_k - 1$  :*

$$C(g' \cdot a_k^{n_k-1}, g \cdot a_k^l) = C(g' \cdot a_k^{n_k-1}, g) \cdot C(g' \cdot a_k^{n_k-1}, a_k)^{l-n_k} \cdot \alpha_{g'}^{n_k-l} \cdot \alpha_g.$$

*If the element  $g$  is written as  $g = a_1^{r_1} a_2^{r_2} \dots a_{k-1}^{r_{k-1}}$  then:*

$$\begin{aligned} \alpha_g &= C(a_1, a_k)^{-r_1 n_k} \dots C(a_{k-1}, a_k)^{-r_{k-1} n_k} \\ \alpha_{a_i} &= C(a_i, a_k)^{-n_k}, \\ C(a_k, g \cdot a_k^{n_k-1}) &= \alpha_g = C(g, a_k)^{-n_k}. \end{aligned}$$

*For  $s \neq n_k - 1$ , where  $\omega_{g,s}^{n_k} = 1$  :*

$$\begin{aligned} C(g \cdot a_k^s, a_k) &= \omega_{g,s} \cdot C(a_1, a_k)^{r_1} \dots C(a_{k-1}, a_k)^{r_{k-1}}, \\ C(g \cdot a_k^{n_k-1}, a_k) &= \omega_g \cdot C(a_1, a_k)^{-r_1(n_k-1)} \dots C(a_{k-1}, a_k)^{-r_{k-1}(n_k-1)} \\ C(a_i, a_k)^{n_i n_k} &= 1. \end{aligned}$$

As in the case of a product of two factors, the  $(1, 2)$ -symmetry provides some extra information about the structure constant  $C$  that we summarize in the following two lemmas. Their proofs follow the same lines as before and will be omitted.

**Lemma 15.** *Suppose that  $G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ , and fix generators  $a_1, a_2, \dots, a_k, a_i \in \mathbb{Z}_{n_i}$ . Suppose that  $W$  is a  $(1, 2)$ -symmetric  $G$ -graded twisted  $\mathbb{C}$ -algebra. If  $\mathcal{B}$  is a standard basis for  $W$  with structure constant  $C : G \times G \rightarrow \mathbb{C}^*$  then:*

$$C(g_1 \cdot a_k^s, g_2 g_3) = C(g_1 \cdot a_k^s, g_2) \cdot C(g_1 \cdot a_k^s, g_3) \cdot C(g_2, g_3)^{-1} \cdot C(g_2, g_1 g_3) \cdot C(g_2, g_1)^{-1}$$

*for every  $g_1, g_2, g_3 \in G_1 = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{k-1}}$ .*

*Furthermore, if  $g = a_1^{r_1} \dots a_i^{r_i} \dots a_{k-1}^{r_{k-1}}$ , and if we denote  $\tilde{g} = a_1^{r_1} \dots a_{i-1}^{r_{i-1}}$ , then*

$$C(g \cdot a_k^s, a_i^j) = C(g \cdot a_k^s, a_i)^j \cdot C(a_i, \tilde{g} \cdot a_i^{r_i})^{-(j-1)} \cdot C(a_i, \tilde{g} \cdot a_i^{r_i+1}) \dots C(a_i, \tilde{g} \cdot a_i^{r_i+j-1}).$$

Hence, if  $r_i = n_i - 1$  :

$$C(g \cdot a_k^s, a_i^j) = C(g \cdot a_k^s, a_i)^j \cdot C(\tilde{g}, a_i)^{(j-1)n_i} \quad \text{if } r_i = n_i - 1.$$

On the other hand, if  $r_i \neq n_i - 1$  :

$$C(g \cdot a_k^s, a_i^j) = C(g \cdot a_k^s, a_i)^j \cdot C(\tilde{g}, a_i)^{-n_i} \quad \text{or} \quad C(g \cdot a_k^s, a_i^j) = C(g \cdot a_k^s, a_i)^j.$$

In particular,

$$C(g \cdot a_k^s, a_i)^{n_i} = C(a_i, \tilde{g} \cdot a_i^{r_i})^{n_i-1} \cdot C(a_i, \tilde{g} \cdot a_i^{r_i+1})^{-1} \cdots C(a_i, \tilde{g} \cdot a_i^{r_i+n_i-1})^{-1}.$$

Therefore,

$$\begin{aligned} C(g \cdot a_k^s, a_i)^{n_i} &= (C(\tilde{g}, a_i)^{-(n_i-1)})^{n_i} \quad \text{if } r_i = n_i - 1, \\ C(g \cdot a_k^s, a_i)^{n_i} &= C(\tilde{g}, a_i)^{n_i} \quad \text{if } r_i \neq n_i - 1. \end{aligned}$$

**Remark 16.** From the Theorem 14 we know that

$$C(\tilde{g}, a_i) = \omega_{\tilde{g}} \cdot C(a_1, a_i)^{r_1} \cdots C(a_{i-1}, a_i)^{r_{i-1}}, \quad \text{where } \omega_{\tilde{g}}^{n_i} = 1.$$

Therefore, the last lemma implies that

$$C(g \cdot a_k^s, a_i)^{n_i} = (C(a_1, a_i)^{r_1} \cdot C(a_{i-1}, a_i)^{r_{i-1}})^{-(n_i-1)n_i}, \quad \text{if } r_i = n_i - 1,$$

and that

$$C(g \cdot a_k^s, a_i)^{n_i} = (C(a_1, a_i)^{r_1} \cdot C(a_{i-1}, a_i)^{r_{i-1}})^{n_i}, \quad \text{if } r_i \neq n_i - 1.$$

Hence,

$$\begin{aligned} C(g \cdot a_k^s, a_i) &= \omega_{g,s} C(a_1, a_i)^{r_1} \cdot C(a_{i-1}, a_i)^{r_{i-1}} \quad \text{if } r_i \neq n_i - 1, \quad \text{where } \omega_{g,s}^{n_i} = 1, \\ C(g \cdot a_k^s, a_i) &= \omega_{g,s} \cdot (C(a_1, a_i)^{r_1} \cdot C(a_{i-1}, a_i)^{r_{i-1}})^{-(n_i-1)}, \quad \text{if } r_i = n_i - 1, \quad \text{where } \omega_{g,s}^{n_i} = 1. \end{aligned} \tag{28}$$

**Lemma 17.** Let  $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ , and fix generators  $a_1, a_2, \dots, a_k$ ,  $a_i \in \mathbb{Z}_{n_i}$ . Suppose that  $W$  is a  $(1, 2)$ -symmetric  $G$ -graded twisted  $\mathbb{C}$ -algebra. Then the constants  $C(g \cdot a_k^s, g')$  always can be expressed in terms of the constants  $C(g \cdot a_k^s, a_i)$ ,  $1 \leq i \leq k-1$  and  $C(a_j, a_t)$ ,  $1 \leq j, t \leq k-1$ , for every  $g, g' \in G_1 = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{k-1}}$ .

The following theorem summarizes the above discussion:

**Theorem 18.** Let  $G$  be presented as  $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ , and fix generators  $a_1, a_2, \dots, a_k$ ,  $a_i \in \mathbb{Z}_{n_i}$ . Suppose that  $W$  is a  $(1, 2)$ -symmetric  $G$ -graded twisted  $\mathbb{C}$ -algebra. If  $\mathcal{B}$  is a standard basis for  $W$  with structure constant  $C : G \times G \rightarrow \mathbb{C}^*$ , then the values of  $C(g \cdot a_k^s, a_i)$ ,  $C(g \cdot a_k^s, a_k)$  and  $C(a_t, a_j)$ , with  $1 \leq i, t, j \leq k-1$ ,  $g \in G_1 = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{k-1}}$  completely determine the structure constant  $C$ . Furthermore, the following identities generalize the ones obtained in (19):

If  $0 \leq l < n_k - s$ , then

$$C(g \cdot a_k^s, g' \cdot a_k^l) = C(g \cdot a_k^s, g') \cdot C(g \cdot a_k^s, a_k)^l.$$

If  $n_k - s \leq l \leq n_k - 1$ ,  $s \neq 0$ ,  $s \neq n_k - 1$ , then

$$C(g \cdot a_k^s, g' \cdot a_k^l) = C(g \cdot a_k^s, g') \cdot C(g \cdot a_k^s, a_k)^{l-n_k} \cdot \alpha_{g'}.$$

If  $s = n_k - 1$  then

$$C(g \cdot a_k^{n_k-1}, g' \cdot a_k^l) = C(g \cdot a_k^{n_k-1}, g') \cdot C(g \cdot a_k^{n_k-1}, a_k)^{l-n_k} \cdot \alpha_g^{n_k-l} \cdot \alpha_{g'}.$$

If  $g$  is written as  $g = a_1^{r_1} a_2^{r_2} \dots a_{k-1}^{r_{k-1}}$  then

$$\begin{aligned} \alpha_g &= C(a_1, a_k)^{-r_1 n_k} \dots C(a_{k-1}, a_k)^{-r_{k-1} n_k} \\ \alpha_{a_i} &= C(a_i, a_k)^{-n_k} \\ C(a_i, a_k)^{n_i n_k} &= 1. \\ C(a_k, g \cdot a_k^{n_k-1}) &= \alpha_g = C(g, a_k)^{-n_k} \\ C(g \cdot a_k^{n_k-1}, a_k) &= \omega_g \cdot C(a_1, a_k)^{-r_1(n_k-1)} \dots C(a_{k-1}, a_k)^{-r_{k-1}(n_k-1)}, \text{ where } \omega_g^{n_k} = 1. \end{aligned}$$

For the case where  $s \neq 0$  and  $g \neq a_j$ , for  $j = 1, 2, \dots, k$ :

$$C(g \cdot a_k^s, a_k) = \omega_{g,s} \cdot C(a_1, a_k)^{r_1} \dots C(a_{k-1}, a_k)^{r_{k-1}}, \text{ for } s \neq n_k - 1, \text{ where } \omega_{g,s}^{n_k} = 1.$$

For the case  $s \neq 0$  and  $g \neq a_j$  for  $j = 1, 2, \dots, k-1$ , and  $i = 1, 2, \dots, k-1$  we have:

$$\begin{aligned} C(g \cdot a_k^s, a_i) &= \omega_{g,s} C(a_1, a_i)^{r_1} \dots C(a_{i-1}, a_i)^{r_{i-1}} \text{ if } r_i \neq n_i - 1, \text{ where } \omega_{g,s}^{n_i} = 1. \\ C(g \cdot a_k^s, a_i) &= \omega_{g,s} \cdot (C(a_1, a_i)^{r_1} \dots C(a_{i-1}, a_i)^{r_{i-1}})^{-(n_i-1)}, \text{ if } r_i = n_i - 1, \text{ where } \omega_{g,s}^{n_i} = 1. \end{aligned}$$

Before starting our next theorem we notice that as in Theorem 9 before two  $(1, 2)$ -symmetric  $\mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_k}$ -graded twisted algebras are graded-isomorphic if and only if they have the same structure constants in their respective standard bases. The following theorem generalizes Theorem 10 when  $G$  is a product of an arbitrary number of cyclic groups.

**Theorem 19.** Let  $G = \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_k}$  and let  $A \subset \mathbb{C}^*$  be a finite subgroup. Suppose that we choose values in  $A$  for  $C(g \cdot a_k^s, a_k)$ ,  $C(a_i, a_j)$  and  $C(g \cdot a_k^s, a_i)$  satisfying the identities in Theorem 18. Then  $W = \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_k}$  with the multiplication given by  $C$  (referred to the canonical basis of  $\mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_k}$ ) is a  $(1, 2)$ -symmetric  $G$ -graded twisted  $\mathbb{C}$ -algebra.

Finally, we may state our main theorem:

**Theorem 20.** The number of non-(graded) isomorphic  $(1, 2)$ -symmetric  $G = \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_k}$ -graded twisted  $\mathbb{C}$ -algebras with structure constants taking values in a finite subgroup  $A \subset \mathbb{C}^*$  is given by the product:

$$\prod_{i=1}^k |R_{n_i}|^{|G|-(k+1)} \prod_{1 \leq i < j \leq k} |R_{n_i n_j}|,$$

where  $R_{n_i}$  denotes the set  $\{\omega \in A : \omega^{n_i} = 1\}$ .

We have given a classification of a concrete family of twisted group algebras satisfying a particular property in their function  $r$ . The diverse properties that can be imposed to the function  $r$ , and to the function  $q$ , lead to further families of algebras which deserve as well classification endeavors. In a forth coming publication we will tackle the relation of these properties and more standard families of algebras such as the diverse sorts of Lie-admissible algebras. In this sense, the approach of studying twisted group algebras via the properties of their  $r$  and  $q$  functions, provide novel powerful tools to classify concrete subclasses of standard algebras such as left-symmetric, pre-Lie, anti-flexible, and further varieties of algebras.



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